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Vacuum polarization in laser fields

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Abstract. According to quantum electrodynamics the vacuum shows polarization properties because of the presence of virtual electron–positron pairs. These properties are investigated in the presence of an intense plane wave field, such as is produced by a laser. The laser wave is considered as an external prescribed field and its interaction with the electron–positron field is treated without reference to perturbation theory. The vacuum polarization tensor is computed to second order in the fine structure constant in the form of a double integral. The Dyson equation for the photon propagator is solved by an eigenfunction expansion. For a plane laser wave of infinite extent and circular polarization the results are relatively simple and explicit. Analytical properties of the photon propagator are discussed. The effects of vacuum polarization on an additional weak wave field (a non-laser photon) can be described approximately by two complex indices of refraction. The corresponding dispersion curves resemble qualitatively those of ordinary optical media. Quantitatively, however, the effects are small and will show up only if the non-laser photon has a very high energy. If the laser frequency could be raised into the x-ray region, the effects could be observed at moderate photon energies.

1. Introduction

Quantum electrodynamics in strong external fields has found renewed interest of late, mainly for two reasons: (i) from the experimental point of view, it may be possible that fields which are 'strong' in the sense of quantum electrodynamics can be found eg on pulsars or near the surface of heavy nuclei; (ii) as for the theory it may be interesting to see what happens if an expansion parameter becomes large. The rapid increase in available laser intensities suggests the consideration of lasers as sources of strong fields. While the simple processes of quantum electrodynamics (Compton scattering, pair creation etc) have been extensively investigated (Eberly 1969, this review paper contains references up to 1968; see also Denisov and Fedorov 1967, Oleinik 1967, Ehlotzky 1970a, b, Reiss 1970, 1971, 1972), relatively little is known about radiative corrections implied by the structure of the vacuum in quantum electrodynamics. Here we concentrate on vacuum polarization and the photon propagator. An eigenvector representation for both objects yields information on the dispersion laws of the medium 'vacuum plus laser field' which govern the propagation of photons (§ 3). Similar investigations for constant fields (Newton 1954a, b, Minguzzi 1956, 1957, 1958, Baier and Breitenlohner 1967a, b, Narozhnyi 1968, Batalin and Shabad 1968, 1971, Shabad 1971, Adler 1971), in particular for crossed fields (Ritus 1969, 1972a, b, Morozov and Ritus 1975) have been carried out in the past. We shall argue, however, that the approximation by a constant crossed field is not allowed in the region where the effects of vacuum polarization become measurable. Some partial results for laser fields have been obtained by other authors before (Oleinik 1967, Yakovlev 1966).

Next we give a representation of the vacuum polarization tensor as a double integral in the general case (§ 4). For an infinite plane wave train of circular polarization this can be further reduced. Due to a special symmetry (Richard 1972) the structure of the propagator is very simple in this case: besides the usual diagonal part there are only two contributions, which correspond to a frequency change by two units (in terms of the laser frequency) accompanied by a helicity flip. The analytical structure shows (complex) poles and some traces of threshold behaviour (§ 5). This structure differs from the corresponding one found for constant crossed fields.

It is well known that the vacuum of quantum electrodynamics behaves in some sense like a nonlinear dielectric medium. Some of these aspects are contained in the vacuum polarization tensor and/or the photon propagator as studied here. Among the numerous physical implications we shall consider only the superposition of a weak wave field onto the laser field. Because of vacuum polarization a linear superposition principle does not apply: the wave field will see the polarization of virtual pairs influenced by the laser. This effect is of second order in the elementary charge e. In addition there are higher-order terms ($\sim e^3, e^4$ etc) which contain higher powers of the wave field and allow for a fusion or fission of (non-laser) quanta in the presence of the laser. If these terms are neglected, we obtain a linear theory for a 'medium' consisting of the vacuum plus the laser field. Maxwell's equations can be solved and it is possible to formulate approximate dispersion laws (§ 6). Unfortunately the magnitude of the effects is not sufficient to make them observable at present.

2. Description of the laser field, symmetries

The laser field will be described by a plane wave of wavelength λ (wavevector k_μ, $k^0 = 2\pi/\lambda$) characterized by the vector potential

$$A_\mu^{(\mathrm{L})}(x) = ae_{i,\mu}a_i(\xi). \tag{2.1}$$

Here $e_{i,\mu}$ ($i = 1, 2$) are two polarization vectors orthogonal to k_μ

$$(k, e_i) = 0$$

and we use the summation convention for repeated polarization indices; $a_i(\xi)$ is a certain function of its argument $\xi = (k, x)$ (see below) and the amplitude factor a is determined by the intensity of the laser beam. We shall consider $A_\mu^{(\mathrm{L})}$ as a classical external field, which is an excellent approximation in this context (Brown and Kibble 1964).

The field tensor of the laser field is

$$F_{\mu\nu}^{(\mathrm{L})}(x) = af_{i,\mu\nu}f_i(\xi) \tag{2.2}$$

where we have

$$f_i(\xi) = \frac{\mathrm{d}a_i(\xi)}{\mathrm{d}\xi}, \qquad f_{i,\mu\nu} = k_\mu e_{i,\nu} - k_\nu e_{i,\mu}. \tag{2.3}$$

The dual tensor is

$$f_{i,\mu\nu}^* = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}f_i^{\alpha\beta} = \epsilon_{ij}f_{j,\mu\nu}, \qquad \epsilon_{ij} = -\epsilon_{ji}, \qquad \epsilon_{12} = 1 \tag{2.4}$$

and we have the relations

$$k^\mu f_{i,\mu\nu} = 0, \qquad f_i^{\mu\alpha} f_j^{*}{}_{\alpha}{}^{\nu} = -k^\mu k^\nu \epsilon_{ij}$$
$$f_i^{\mu\alpha} f_{j\alpha}{}^{\nu} = f_i^{*\mu\alpha} f_{j\alpha}^{*}{}^{\nu} = k^\mu k^\nu \delta_{ij}. \tag{2.5}$$

The Poynting vector of the laser field is

$$|\boldsymbol{S}| = \frac{a^2 \pi c}{\lambda^2} f_i(\xi) f_i(\xi).$$

From this we can compute in practical cases the magnitude of effects. An appropriate expansion parameter is the (classical, dimensionless) quantity

$$\nu^2 = \left(\frac{ea}{m_e c^2}\right)^2 = \left(\frac{\epsilon a}{\kappa}\right)^2 \tag{2.6}$$

where $e = \hbar c \epsilon$ is the elementary charge and $m_e = \kappa \hbar / c$ is the rest mass of the electron. If the laser is characterized by an illumination per square wavelength I we have

$$\nu^2 \simeq 7 \times 10^{-11} I$$

where I is given in $\mathrm{W}\lambda^2(\mathrm{m})\mathrm{m}^{-2}$. It has to be noted that this parameter is not small for available high-intensity lasers.

It will be convenient to use light-like components (Neville and Rohrlich 1971). For this purpose we introduce a fixed orthogonal *vierbein* in Minkowski space

$$n^\mu = \frac{1}{\omega\sqrt{2}} k^\mu = \frac{1}{\sqrt{2}}(1, \boldsymbol{n}) \qquad \boldsymbol{n} \cdot \boldsymbol{n} = 1, \qquad \omega = k^0 \qquad \hat{n}^\mu = \frac{1}{\sqrt{2}}(1, -\boldsymbol{n})$$
$$e_i^\mu = (0, \boldsymbol{e}_i), \qquad \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \tag{2.7}$$

where we have

$$n^2 = \hat{n}^2 = (n, e_i) = (\hat{n}, e_i) = 0, \qquad (n, \hat{n}) = 1, \qquad (e_i \cdot e_j) = -\delta_{ij}.$$

An arbitrary vector p^μ can be represented according to

$$p^\mu = n^\mu p_u + \hat{n}^\mu p_v + e_i^\mu p_i \tag{2.8}$$

in terms of its 'light-like components'

$$p_u = (\hat{n}, p), \qquad p_v = (n, p), \qquad p_i = -(e_i, p). \tag{2.9}$$

For the coordinate vector we have $\xi = \omega x_v \sqrt{2}$.

The advantage of this formalism (especially with respect to boundary conditions for scattering problems) has been stressed by Neville and Rohrlich. Scattering problems can be defined for laser pulses of finite duration:

$$a_i(\xi) = 0 \qquad \text{for } \xi < \xi_0 \text{ and } \xi > \xi_1 \text{ with fixed } \xi_0, \xi_1. \tag{2.10}$$

As a consequence of translation invariance along the directions x_u, x_i the components P_v, P_i of the total energy momentum vector P_μ are conserved for all reactions involving particles in laser beams.

In general we shall consider finite laser pulses (2.10) without specifying a_i further. Sometimes it will be convenient, however, to consider an infinite plane wave train as an approximation to the case of practical interest, where $\xi_1 - \xi_0$ is large. In this case the a_i are periodic functions of ξ. As a consequence of the lattice-like translational invariance, the component $P_u/(\omega\sqrt{2})$ is then conserved *modulo* an integer (Zeldovich 1966) so that we have

$$P'_\mu - P_\mu = lk_\mu, \qquad l = 0, \pm 1, \pm 2, \ldots.$$

For a circularly polarized plane wave train

$$a_1(\xi) = \cos\xi, \qquad a_2(\xi) = -\sin\xi \tag{2.11}$$

the expressions to be considered will have the simplest form. In this case the problem has an additional symmetry: the external field remains invariant under a translation by an arbitrary vector $\boldsymbol{b}$ plus a rotation about the direction of propagation $\boldsymbol{k}$ by the angle $-(\boldsymbol{k}\,.\,\boldsymbol{b})$.

Finally we want to note that the case of constant crossed fields as considered by Ritus (1969, 1972a, b) can be obtained eg by taking

$$a_1(\xi) = a_2(\xi) = \xi. \tag{2.12}$$

In this case one has to identify $a\omega\sqrt{2}$ with the magnitude of the field strength.

3. Photon propagation

The polarization phenomena, which are induced in the vacuum due to the action of an external field on the virtual electron–positron pairs, can be investigated by means of the photon propagator D', which is a solution of the Schwinger–Dyson equation

$$D'_{\mu\nu}(x, y) = D_{\mu\nu}(x-y) + \epsilon^2 \int \mathrm{d}^4z\, \mathrm{d}^4z'\, D_{\mu\rho}(x-z)\pi^{\rho\sigma}(z, z')D'_{\sigma\nu}(z', y). \tag{3.1}$$

Here D is the free propagator and π is the vacuum polarization tensor. To lowest-order perturbation theory we have

$$\pi_{\mu\nu}(x, y) = -\mathrm{i}\,\mathrm{Tr}\,\gamma_\mu G(x, y)\gamma_\nu G(y, x) + \ldots \tag{3.2}$$

where G is the electron propagator in the presence of the external field. We shall now indicate how equation (3.1) can be resolved for a general external field for given π. We shall consider the Fourier transform

$$\tilde{\pi}_{\mu\nu}(p, p') = \frac{1}{(2\pi)^4} \int \mathrm{d}^4x\, \mathrm{d}^4x'\, \mathrm{e}^{\mathrm{i}(px)}\, \mathrm{e}^{-\mathrm{i}(p'x')}\pi_{\mu\nu}(x, x') \tag{3.3}$$

(correspondingly for D'). Because of gauge invariance we have

$$p^\mu \tilde{\pi}_{\mu\nu}(p, p') = \tilde{\pi}_{\mu\nu}(p, p')p'^\nu = 0. \tag{3.4}$$

In addition π has the symmetry property

$$\tilde{\pi}_{\mu\nu}(p, p') = \tilde{\pi}_{\nu\mu}(-p', -p). \tag{3.5}$$

The latter property may be traced back to the corresponding one for the photon propagator, which is evident in coordinate space.

We define now left and right eigenvectors of

$$P_{\mu\nu}(p, p') = \frac{1}{p'^2}\tilde{\pi}_{\mu\nu}(p, p') \tag{3.6}$$

by

$$\begin{aligned}\int \mathrm{d}^4p' P_{\mu\nu}(p, p')\epsilon^{(i)\nu}(p', q) &= \kappa_i(q)\epsilon_\mu^{(i)}(p, q)\\ \int \mathrm{d}^4p \bar{\epsilon}^{(i)\mu}(p, q)P_{\mu\nu}(p, p') &= \kappa_i(q)\bar{\epsilon}_\nu^{(i)}(p', q).\end{aligned} \tag{3.7}$$

Left and right eigenvectors corresponding to different eigenvalues $\kappa_i(q)$ can be shown to be orthogonal. By means of the symmetry relation (3.5) the left eigenvectors are related to the right ones: if $\epsilon_\mu^{(i)}(p, q)$ is a right eigenvector, then $\epsilon_\mu^{(i)}(-p, q)/p^2$ is a left eigenvector with the same eigenvalue. We can choose the eigenvectors to be orthonormal (provided they are numbered appropriately):

$$\int \mathrm{d}^4p \bar{\epsilon}^{(i)\mu}(p, q)\epsilon_\mu^{(j)}(p, q') = \delta^{(i)(j)}\delta(q-q'). \tag{3.8}$$

We have then the expansion

$$P_{\mu\nu}(p, p') = \sum_i \int \mathrm{d}^4q \kappa_i(q)\epsilon_\mu^{(i)}(p, q)\bar{\epsilon}_\nu^{(i)}(p', q). \tag{3.9}$$

Because of relations (3.4) we need only three pairs of eigenvectors ($i = 1, 2, 3$) which are orthogonal to p or respectively p'.

If we write the transverse part of $\tilde{D}'_{\mu\nu}$ as

$$(\tilde{D}'_{\mu\nu}(p, p'))_{\mathrm{tr}} = \Delta_{\mu\nu}(p, p')/p^2$$

we obtain the following form for Dyson's equation (3.1):

$$\Delta_{\mu\nu}(p, p') = \left(\frac{p_\mu p_\nu}{p^2} - g_{\mu\nu}\right)\delta(p-p') - \epsilon^2 \int \mathrm{d}^4p'' P_{\mu\lambda}(p, p'')\Delta^\lambda{}_\nu(p'', p'). \tag{3.10}$$

By iteration we may show that Δ has the same eigenvectors as P. Therefore we can expand Δ in the same fashion as (3.9). We obtain

$$\tilde{D}'_{\mu\nu}(p, p') = \frac{1}{p^2 + \mathrm{i}0}\left(G\frac{p_\mu p_\nu}{p^2}\delta(p-p') - \sum_{i=1}^{3} \int \frac{\mathrm{d}^4q}{1+\epsilon^2\kappa_i(q)}\epsilon_\mu^{(i)}(p, q)\bar{\epsilon}_\nu^{(i)}(p', q)\right) \tag{3.11}$$

where we have included a longitudinal term which depends on the gauge via the arbitrary constant G. The apparent pole at $p^2 = 0$ may be absent since it may eventually be cancelled by a contribution from a singularity of $\kappa_i(q)$. If the external field is a monochromatic plane wave field, the eigenvectors have contributions proportional to $\delta(p-q+rk)$ with $r = 0, \pm 1, \pm 2, \ldots$. Positive values of r correspond to a 'fusion' of the original photon with r laser quanta, negative values to the opposite process.

The poles of the integrand of $\tilde{D}'$, given by $1+\epsilon^2\kappa_i(q) = 0$, determine the dispersion laws for our system (vacuum plus external field). If another weak external wave field A_μ is superimposed, it has to satisfy Maxwell's equations

$$\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = j_\nu^{(\mathrm{P})} = \epsilon^2 \int \mathrm{d}^4x' \pi_{\nu\lambda}(x, x')A^\lambda(x') + \ldots \tag{3.12}$$

where $j_\nu^{(\mathrm{P})}$ is the polarization current induced by this field in the vacuum. The dots stand for higher-order contributions ($\sim\epsilon^3$ etc) which involve higher powers of A_μ and allow for a fusion or fission of non-laser photons stimulated by the laser field. The corresponding tensor coefficients π contain three or more electron propagators and have thus far not been computed. If these contributions are neglected, equation (3.12) can be solved. The result is in momentum space (up to a gauge term)

$$\tilde{A}_\mu(p) = \frac{1}{p^2}\sum_i \int d^4q\epsilon_\mu^{(i)}(p, q)\delta(1+\epsilon^2\kappa_i(q))h^{(i)}(q) \tag{3.13}$$

with arbitrary functions $h^{(i)}$. The 'photon excitation' of the medium is thus given by the poles of the propagator, as it must be. The nature of the spectrum has, of course, to be determined by calculation of $\pi_{\mu\nu}$ and solution of the eigenvalue equations (3.7). We shall do this in the case of circular polarization below.

4. The vacuum polarization tensor

We shall now evaluate the expression for π, equation (3.2). Several representations of the electron propagator G are given in another paper (Mitter 1975, to be referred to as M). For the calculation of π the form (M 4.3) with (M 4.13, 14) has turned out to be most convenient. The definition of the functions appearing in G and some useful formulae connecting them can be found in appendix 1.

At first we shall concentrate on some general properties of π. From gauge invariance (3.4) we conclude that $\pi_{\mu\nu}$ depends on the laser field only via $f_{i,\mu\nu}$. If we use a perturbation analysis in powers of the laser field for the Green function, we conclude from Furry's theorem that $\pi_{\mu\nu}$ has to be an even function of $f_{i,\mu\nu}$. Since scalar products of $f_{i,\mu\nu}$ can be reduced to k (cf (2.5)), the dependence on the external field may be formulated in terms of the vectors

$$f_\mu^{(i)} = f_{i,\mu\lambda}p^\lambda/(pk), \qquad f'^{(i)}_\mu = f_{i,\mu\lambda}p'^\lambda/(p'k). \tag{4.1}$$

Analysing in terms of light-like components (cf § 2) we have

$$p_v = p'_v, \qquad p_i = p'_i$$

which means that the difference between p^μ and p'^μ has to be proportional to k^μ (thus we have, eg $p^2+p'^2 = 2(pp')$). Therefore we find

$$f_\mu^{(i)} = f'^{(i)}_\mu = (p_in_\mu + p_ve_{i,\mu})/p_v.$$

Since we have, in addition,

$$f_\mu^{(i)}f^{(j)\mu} = -\delta^{ij} \tag{4.2}$$

we have only three independent invariants which we can take to be, eg,

$$(pp'), \qquad (pk), \qquad p^2-p'^2 = 2p_v(p_u-p'_u). \tag{4.3}$$

In this context it is important to note that the only invariant containing the external field tensor

$$p_\mu f_{i,}{}^{\mu\rho}f_{j,\rho\sigma}p^\sigma = (pk)^2\delta_{ij}$$

is quadratic in the field. Introducing instead of p, p' the gauge invariant combinations

$$g_\mu = p_\mu - \frac{p^2}{(pk)}k_\mu, \qquad g'_\mu = p'_\mu - \frac{p'^2}{(pk)}k_\mu \tag{4.4}$$

(which are orthogonal to f^{i}_{μ}) we can write the most general expression for $\pi_{\mu\nu}$, which is consistent with all these requirements, in the following form:

$$\tilde{\pi}_{\mu\nu}(p,p') = \tilde{\pi}^{(0)}_{\mu\nu} + \tilde{\pi}^{(L)}_{\mu\nu} = \delta(p_v - p'_v)\delta(p_i - p'_i)[G_{\mu\nu}(A^{(0)} + A^{(L)}) + f^{(i)}_{\mu} f^{(j)}_{\nu} B^{(L)}_{ij}]. \tag{4.5}$$

Here the suffix (0) denotes the vacuum polarization in absence of the laser field, whereas (L) denotes the contribution from the latter, and we have introduced the tensor

$$G_{\mu\nu} = p'_{\mu}p_{\nu} - (pp')g_{\mu\nu} = g_{\mu}g'_{\nu} + (pp')f^{(i)}_{\mu}f^{(i)}_{\nu}. \tag{4.6}$$

The coefficients A and B_{ij} are functions of our invariants (4.3). The tensor decomposition (4.5) suggests the *ansatz*

$$\epsilon^{(i)}_{\mu}(p,q) = g_{\mu}a^{(i)}(p,q) + f^{(k)}_{\mu}b^{(i)}_{k}(p,q) \tag{4.7}$$

for the right-hand eigenvectors of $P_{\mu\nu}$. The eigenvalue equation (3.7) then reduces to the one-dimensional integral equations

$$\int \mathrm{d}p'_u A(p,p')a^{(i)}(p',q) = -\kappa_i(q)a^{(i)}(p,q) \tag{4.8}$$

$$\int \frac{\mathrm{d}p'_u}{p'^2}((pp')A(p,p')b^{(i)}_{k}(p',q) + B^{(L)}_{kl}(p,p')b^{(i)}_{l}(p',q)) = -\kappa_i(q)b^{(i)}_{k}(p,q) \tag{4.9}$$

where $A = A^{(0)} + A^{(L)}$, $p' = \{p'_u, p_v, p_i\}$. Obviously, we have three types of solution:

$$\epsilon^{(1)}_{\mu}(p,q) = g_{\mu}a(p,q) \qquad \text{with eigenvalue } \kappa_1(q)$$

$$\epsilon^{(2,3)}_{\mu}(p,q) = b^{(2,3)}_{k}(p,q)f^{(k)}_{\mu} \qquad \text{with eigenvalues } \kappa_{2,3}(q).$$

We shall now turn to the actual evaluation of these functions. Since the details of the computation are rather tedious, we shall give only an outline. We start with the representation

$$\pi_{\mu\nu}(x,x') = \frac{1}{(2\pi)^4}\int \mathrm{d}^4q\, \mathrm{e}^{-\mathrm{i}(q,x-x')}\pi_{\mu\nu}(q|\xi,\xi') \tag{4.10}$$

and consider at first the quantity

$$\pi_{\mu\nu}(q|\xi,\xi') = \frac{-\mathrm{i}}{(2\pi)^4}\int \mathrm{d}^4q'\ \mathrm{Tr}\,\gamma_{\mu}\tilde{G}(q'+\tfrac{1}{2}q|\xi,\xi')\gamma_{\nu}\tilde{G}(q'-\tfrac{1}{2}q|\xi',\xi) \tag{4.11}$$

which can be evaluated by means of the standard techniques used in perturbation theory. If the expression (M 4.14) for the Green function is inserted, we obtain four terms, according to the two contributions from the Green function. Using symmetry properties, we can argue that only the product of the first two terms is divergent and needs regularization. We use the gauge invariant analytic regularization method as described elsewhere (Breitenlohner and Mitter 1968) for the divergent contributions, which applies also here in spite of the fact that the mass m^2 occurring in the denominators is now space-dependent and that different masses appear in the numerator and denominators: we may for instance rewrite the numerator by means of equation (M 4.10) in terms of m^2 and T and use the former quantity as a mass parameter in the spirit of that reference. The result is

$$\pi_{\mu\nu}(q|\xi,\xi') = (q_{\mu}q_{\nu} - q^2 g_{\mu\nu})\pi^{(0)}(q^2) + \pi^{(L)}_{\mu\nu}(q|\xi,\xi'). \tag{4.12}$$

The first term is just the vacuum polarization in absence of the laser field:

$$\pi^{(0)}(q^2) = \frac{1}{4\pi^2}\left[-\frac{C}{3}+\frac{1}{2}\int_0^1 \mathrm{d}y(1-y^2)\ln\left(1-\frac{q^2(1-y^2)}{4\kappa^2}\right)\right]. \tag{4.13}$$

Here C is an arbitrary finite constant which is fixed by charge renormalization. If we want to identify e with the physical charge, we have to take $\pi^{(0)}(0) = 0$, ie $C = 0$. For the other terms we obtain

$$\begin{aligned}\pi^{(\mathrm{L})}_{\mu\nu}(q|\xi,\xi') = \frac{1}{8\pi^2}\int_0^1 \mathrm{d}y\Bigg(&2g_{\mu\nu}T\ln\frac{Q}{\kappa^2}+(1-y^2)(q_\mu q_\nu-q^2g_{\mu\nu})\ln\frac{Q}{Q_0}+\frac{1-y^2}{Q}2k_\mu k_\nu K_{ii}\\ &+\frac{1+y^2}{Q}[\mathrm{i}R(k_\mu q_\nu+k_\nu q_\mu-(qk)g_{\mu\nu})-\hat{R}\epsilon_{\mu\nu\alpha\beta}k^\alpha q^\beta]\\ &+\frac{1-y^2}{Q^2}k_\mu k_\nu(4\kappa^2M_iM_i+R^2-\hat{R}^2)-\frac{(1-y^2)^2}{Q^2}[\tfrac{1}{2}g_{\mu\nu}(qk)^2K_{ii}\\ &+(q_\mu k_\nu-q_\nu k_\mu)(qk)L_iM_i+q^\rho q^\sigma f_{i,\mu\rho}f_{j,\nu\sigma}K_{ij}]\Bigg).\end{aligned} \tag{4.14}$$

Here we have used the abbreviations

$$K_{ij} = L_iL_j+\epsilon_{ik}\epsilon_{jl}M_kM_l \tag{4.15a}$$

$$Q = m^2(\xi,\xi')-\tfrac{1}{4}(1-y^2)q^2. \tag{4.15b}$$

Q_0 denotes the same expression with m^2 replaced by κ^2, its value for $\zeta = \xi-\xi' = 0$.

This has now to be inserted into equation (4.10) and the momentum transform (3.3) has to be evaluated. This is trivial for the free term, which contributes only to A in equation (4.5). We obtain

$$A^{(0)}(p,p') = \delta(p_u-p_u')\pi^{(0)}(p^2). \tag{4.16}$$

The contributions from the laser field $A^{(\mathrm{L})}$ and $B^{(\mathrm{L})}_{ij}$ are most conveniently evaluated using light-like components. These are most natural in this context and simplify the calculation considerably, since most of the integrations turn out to be trivial. The non-trivial integrations involve

$$u = \frac{1}{\omega\sqrt{2}}\zeta, \qquad u' = \frac{1}{\omega\sqrt{2}}\eta, \qquad q_u, \qquad y.$$

It is convenient to introduce the variable

$$\lambda = q_u-\tfrac{1}{2}(p_u+p_u')$$

instead of q_u and to perform at first the Fourier integral on λ, which can be done by means of the standard form

$$\int_{-\infty}^{+\infty}\frac{\mathrm{d}\lambda}{Q}\mathrm{e}^{-\mathrm{i}u\lambda} = \frac{4\pi\mathrm{i}\Theta(up_v)}{|p_v|(1-y^2)}\exp\left[\frac{\mathrm{i}u}{2p_v}\left((pp')-\frac{4m^2}{1-y^2}\right)\right].$$

All other types of integral appearing in the calculation can be derived from this formula by differentiation and/or partial integration with respect to the other variables. These partial integrations can also be used in order to rewrite the polarization tensor in such a way that gauge invariance becomes manifest. It has to be observed that the boundary terms encountered in these integrations are zero for a laser field of finite duration (for a field of infinite extent one has to worry about rapidly oscillating contributions, whose role is somewhat obscure). As a result of these manipulations we obtain

$$\begin{aligned} A^{(\mathrm{L})}(p, p') &= \frac{1}{2\pi} \int \mathrm{d}u' \, \mathrm{e}^{\mathrm{i}u'(p_u - p'_u)}(K^{(1)} - K^{(1)}|_{m^2 \to \kappa^2}) \\ B^{(\mathrm{L})}_{ij}(p, p') &= \frac{1}{2\pi} \int \mathrm{d}u' \, \mathrm{e}^{\mathrm{i}u'(p_u - p'_u)}(\delta_{ij} K^{(2)} + 2K^{(3)}_{ij} + \epsilon_{ij} \operatorname{sgn}(pk) K^{(4)}). \end{aligned} \tag{4.17}$$

The coefficients can be written as Fourier integrals

$$K^{(l)}((pp'), u') = \frac{1}{4\pi^2} \int_0^\infty \mathrm{d}\zeta \int_0^1 \mathrm{d}y P^{(l)}(\zeta, u', y) \exp\left[\frac{\mathrm{i}\zeta}{2|(pk)|}\left((pp') - \frac{4m^2}{1-y^2}\right)\right] \tag{4.18}$$

where we have

$$\begin{aligned} P^{(1)} &= -\frac{1-y^2}{2\zeta}, & P^{(2)} &= R + N - \frac{2R}{1-y^2} \\ P^{(3)}_{ij} &= \zeta K_{ij}, & P^{(4)} &= -\mathrm{i}\hat{R}\left(\frac{2}{1-y^2} - 1\right). \end{aligned} \tag{4.19}$$

From the symmetry properties of the $P^{(j)}$ we may check that the relation (3.4) is fulfilled.

The vectors (4.1) and (4.4) chosen for the decomposition (4.5) are singular if p or p' is parallel to k. In this case direct computation from equation (4.14) shows that

$$\pi^{(\mathrm{L})}_{\mu\nu}(p, p') = 0 \qquad \text{if } p_v = p_i = 0. \tag{4.20}$$

For constant crossed fields (2.12) the Fourier integral on u' turns out to be trivial and one may obtain the results given by Ritus (1972a, b) for the polarization tensor.

In the general case the integral on y can be performed. Transforming

$$\alpha + 1 = \frac{1}{1-y^2}$$

and using the formula

$$\int_0^\infty \frac{\mathrm{d}\alpha}{[\alpha(\alpha+\beta)]^{1/2}} \mathrm{e}^{-\mathrm{i}\alpha z} = \frac{\pi}{2\mathrm{i}} \mathrm{H}^{(2)}_0\left(\frac{\beta z}{2}\right) \mathrm{e}^{\mathrm{i}\beta z/2}$$

we can express the coefficients in terms of Hankel functions of a single variable. It is convenient to transform $\zeta = \rho t$ where

$$\rho = \frac{2|(pk)|}{\kappa^2}. \tag{4.21}$$

The argument of the Hankel function is then

$$w = 2\frac{m^2}{\kappa^2} t \tag{4.22}$$

and we have

$$K^{(l)}((pp'), u') = \frac{\mathrm{i}\rho}{8\pi} \int_0^\infty \mathrm{d}t B^{(l)}(\rho t, u') \exp\left(\frac{\mathrm{i}t}{\kappa^2}((pp') - 2m^2)\right) \tag{4.23}$$

$$
\begin{aligned}
B^{(1)} &= \frac{2m^2\rho}{3\kappa^2}(2\mathrm{i}S(w)+\mathrm{H}_1^{(2)}(w)) \\
B^{(2)} &= R\mathrm{H}_0^{(2)}(w)+(R+N)S(w) \\
B_{ij}^{(3)} &= \rho t S(w) K_{ij} \\
B^{(4)} &= \mathrm{i}\hat{R}(\mathrm{H}_0^{(2)}(w)+S(w)) \\
S(w) &= w(\mathrm{i}\mathrm{H}_0^{(2)}(w)-\mathrm{H}_1^{(2)}(w)).
\end{aligned}
\tag{4.24}
$$

Here the quantities m^2, R, $\hat{R}$, N are functions of the arguments

$$\zeta = \rho t \qquad \text{and} \qquad \eta = \omega u'\sqrt{2}\,.$$

By expanding the functions M_i, L_i etc we can infer that for small ρ the leading contributions to $\pi_{\mu\nu}^{(\mathrm{L})}$ are of the order $v^2\rho^2$. In cases of practical interest ρ^2 is a small parameter, even if we take for p the momentum of a photon in the GeV region. Therefore we expect the effects of $\epsilon^2\pi_{\mu\nu}^{(\mathrm{L})}$ to be very small, even if the laser intensity is high ($v^2 \sim 1$). In addition we see that equation (4.20) holds for $p_v = 0$ alone, since $\pi_{\mu\nu}^{(\mathrm{L})}$ tends to zero for $\rho \to 0$.

5. Results for circular polarization

For the circular polarized plane wave (2.11) the functions A and B_{ij} simplify considerably, since among the functions (4.24) only $B_{ij}^{(3)}$ depends on the centre of mass coordinate u' (see appendix 1). The integration can be performed. Retaining only the first argument of the quantities $K^{(i)}$ introduced in equation (4.18) and using the abbreviations $K_{ii}^{(3)} \equiv K^{(3)}$, $p_\pm = p \pm 2k$, $\beta_{11} = -\beta_{22} = -\mathrm{i}\beta_{12} = -\mathrm{i}\beta_{21} = 1$, we have

$$
\begin{aligned}
A(p,p') &= \delta(p_u-p'_u)(\pi^{(0)}(p^2)+K^{(1)}(p^2)-K^{(1)}(p^2)|_{m^2\to\kappa^2}) \;=\; :\delta(p_u-p'_u)A(p^2) \\
B_{ij}^{(\mathrm{L})}(p,p') &= \delta(p_u-p'_u)(\delta_{ij}(K^{(2)}(p^2)+K^{(3)}(p^2))+\epsilon_{ij}\,\mathrm{sgn}(pk)K^{(4)}(p^2)) \\
&\quad +\tfrac{1}{2}\delta(p_{+,u}-p'_u)\beta_{ij}K^{(3)}(pp_+)+\tfrac{1}{2}\delta(p_{-,u}-p'_u)\beta_{ij}^*K^{(3)}(pp_-).
\end{aligned}
\tag{5.1}
$$

All the invariant functions depend in addition on ρ. We have suppressed this variable, which is a constant of motion.

The solution of equation (4.8) is simply

$$\epsilon_\mu^{(1)}(p,q) = g_\mu a(q)\,\delta(p-q), \qquad \kappa_1(q) = -A(q^2). \tag{5.2}$$

The equations (4.9) can be simplified by means of the combinations

$$b_\pm^{(2,3)}(p,q) = \frac{1}{\sqrt{2}}(b_1^{(2,3)} \pm \mathrm{i}b_2^{(2,3)})$$

and take the form

$$\left(\frac{1}{p^2}N^{(\mp)}(p)+\kappa_i(q)\right)b_\pm^{(i)}(p,q)+\frac{1}{p_\mp^2}K^{(3)}(pp_\mp)b_\mp^{(i)}(p_\mp,q) = 0 \tag{5.3}$$

where

$$N^{(\pm)}(p) = p^2A(p^2)+K^{(2)}(p^2)+K^{(3)}(p^2)\pm\mathrm{i}\,\mathrm{sgn}(pk)K^{(4)}(p^2). \tag{5.4}$$

Because of the last term, $N^{(\pm)}(p)$ and also $\kappa_{2,3}(p)$ depend on the sign of p and we have $N^{(\pm)}(-p) = N^{(\mp)}(p)$. A solution of equation (5.3) is obtained by the *ansatz*

$$b_+^{(i)}(p,q) = \beta_+^{(i)}(q)\delta(p_- - q), \qquad b_-^{(i)}(p,q) = \beta_-^{(i)}(q)\delta(p-q) \tag{5.5}$$

and we obtain for the eigenvalues

$$\kappa_{2,3}(q) = -\frac{1}{2}\left(\frac{N^{(+)}(q)}{q^2}+\frac{N^{(-)}(q_+)}{q_+^2}\right) \pm \left[\frac{1}{4}\left(\frac{N^{(+)}(q)}{q^2}-\frac{N^{(-)}(q_+)}{q_+^2}\right)^2 + \frac{(K^{(3)}(qq_+))^2}{q^2q_+^2}\right]^{1/2}. \tag{5.6}$$

In terms of

$$f_\mu = \frac{1}{\sqrt{2}}(f_\mu^{(1)} + \mathrm{i} f_\mu^{(2)})$$

the eigenvectors read

$$\epsilon_\mu^{(i)}(p,q) = f_\mu \beta_-^{(i)}(q)\delta(p-q) + f_\mu^* \delta(p_- - q)\beta_+^{(i)}(q) \tag{5.7}$$

and we have the relation

$$\begin{aligned}\beta_+^{(i)}(q) &= -\frac{q_+^2}{K^{(3)}(qq_+)}\left(\frac{N^{(+)}(q)}{q^2}+\kappa_i(q)\right)\beta_-^{(i)}(q)\\ &= -\frac{K^{(3)}(qq_+)}{q^2}\left(\frac{N^{(-)}(q_+)}{q_+^2}+\kappa_i(q)\right)^{-1}\beta_-^{(i)}(q).\end{aligned} \tag{5.8}$$

We shall not discuss the consequences of the normalization, but remark only that the relation between left and right eigenvectors has to be handled with care, since the eigenvalues are doubly degenerate:

$$\kappa_1(q) = \kappa_1(-q), \qquad \kappa_{2,3}(q) = \kappa_{2,3}(-q_+).$$

The orthonormality relation (3.8) is fulfilled if we identify

$$\bar{\epsilon}_\mu^{(1)}(p,q) = \frac{1}{p^2}\epsilon_\mu^{(1)}(-p,-q)$$

$$\bar{\epsilon}_\mu^{(2,3)}(p,q) = \frac{1}{p^2}\epsilon_\mu^{(2,3)}(-p,-q_+).$$

The representation (3.11) of the propagator can be evaluated with the result

$$\begin{aligned}\tilde{D}'_{\mu\nu}(p,p') = {}& \frac{\delta(p-p')}{p^2+\mathrm{i}0}\left(G\frac{p_\mu p_\nu}{p^2}+\frac{g_\mu g_\nu}{p^2}\frac{1}{1-\epsilon^2 A(p^2)}\right)\\ &+\delta(p-p')\left(f_\mu f_\nu^*\frac{p_+^2-\epsilon^2N^{(-)}(p_+)}{\Delta(p)}+f_\mu^* f_\nu\frac{p_-^2-\epsilon^2N^{(+)}(p_-)}{\Delta(p_-)}\right)\\ &+\delta(p-p'+2k)f_\mu f_\nu\frac{\epsilon^2K^{(3)}(pp_+)}{\Delta(p)}+\delta(p-p'-2k)f_\mu^* f_\nu^*\frac{\epsilon^2K^{(3)}(pp_-)}{\Delta(p_-)}\end{aligned} \tag{5.9}$$

where we have used the abbreviation

$$\begin{aligned}\Delta(p) = \Delta(-p_+) &= p_+^2p^2(1+\epsilon^2\kappa_2(p))(1+\epsilon^2\kappa_3(p))\\ &= (p^2-\epsilon^2N^{(+)}(p))(p_+^2-\epsilon^2N^{(-)}(p_+))-\epsilon^4(K^{(3)}(pp_+))^2.\end{aligned} \tag{5.10}$$

From a naive consideration of Dyson's equation one might have guessed that the slight non-diagonality of $\tilde{\pi}_{\mu\nu}$ (which contains terms of the order $\delta(p-p'\pm 2k)$) expands to arbitrarily high non-diagonal terms of the order $\delta(p-p'\pm 2rk)$. That these terms are absent and the propagator is relatively simple is due to the additional symmetry for circular polarization as mentioned in § 2. In fact the essential properties of expression (5.9)—the conservation laws for p and p' corresponding to the various basic tensors—can be derived from this symmetry without explicit calculation and remain valid to all orders of ϵ^2. The argument is given in appendix 2. Thus the only effect of higher contributions to $\pi_{\mu\nu}$ is to modify the functions $N^{(\pm)}$ and $K^{(3)}$ by terms of higher order in ϵ^2. The information which the propagator contains on the electromagnetic behaviour of our medium is displayed in the matrix elements for physical processes described by Feynman diagrams, which contain the propagator (eg Møller scattering). Since the quantities (4.23) are complex functions, the analytical structure of $\tilde{D}'_{\mu\nu}$ is important for these matrix elements. We shall discuss some of its aspects briefly.

The pole structure of the propagator (5.9) is of main interest. The pole of the second term at $p^2 = 0$ is evident. This pole does not correspond to photon type excitations here, since g_μ becomes purely longitudinal at $p^2 = 0$. The term will however influence the potential between two charged particles in the medium. The other terms have apparently no poles at $p^2 = 0$, as can be seen from the second form (5.10). If we neglect the last term in this form (which is of the order ϵ^4), Δ factorizes and one of the factors is cancelled by the numerator in each of the two transverse contributions to the diagonal part of the propagator. Thus the poles in these terms are approximately given by

$$p^2 = \epsilon^2 N^{(\mp)}(p) + \mathrm{O}(\epsilon^4) \tag{5.11}$$

corresponding to excitations with left and right helicity respectively. In all cases of practical interest we have $\nu^2 \lesssim 1$ and $\rho \lesssim 1$. From (4.23) and (A1.7) we may then conclude that $N^{(\pm)}$ is of the order $\nu^2\rho^2$. Thus we can replace condition (5.11) approximately by

$$p^2 \simeq \epsilon^2 N^{(\mp)}(0) + \mathrm{O}(\epsilon^4) \tag{5.12}$$

where $N(0)$ means $N(p)|_{p^2=0}$. The argument leading to (5.12) breaks down if $\epsilon^2\nu^2\rho^2 \gg 1$. In practice the formula may however be used even for relatively large values of ρ.

The dispersion laws, which correspond to the poles (5.12), will be studied in § 6. Both poles reduce to $p^2 = 0$ if the laser is absent. The non-diagonal terms are smaller by a relative factor ϵ^2. Each of them changes the helicity and contains both poles.

It is possible that there is a large number of other poles. If, for instance, the functions $\kappa_i(p)$ exhibit an essential singularity at $p^2 = \infty$ (as in the case of constant, crossed fields (Ritus 1972a)), there must be infinitely many poles in the vicinity of that point due to Picard's theorem. Now all three $\kappa_i(p)$ can be shown to decrease at least as $(p^2)^{-1}$ as $p^2 \to \pm\infty$ along the real axis. So, with some smoothness assumptions, additional poles can only occur far from the real axis, if they exist at all. They would then correspond to rapidly decaying modes without much physical significance.

Some further information about the analytical properties of the propagator can be obtained by investigating the corresponding behaviour of the functions $K^{(i)}$. We use the representation (4.23) and consider the convergence properties for large t and fixed ρ. Since the variable w of (4.22) is large in this limit, we may use the asymptotic expansion of the Hankel functions

$$\mathrm{H}_\nu^{(2)}(w)\exp\left(\frac{\mathrm{i}t}{\kappa^2}(p^2-2m^2)\right) \sim w^{-1/2}\exp\left(\frac{\mathrm{i}t}{\kappa^2}(p^2-4m^2)\right)[1+\mathrm{O}(w^{-1})].$$

If we expand the exponential according to

$$\exp \frac{it}{\kappa^2}(p^2 - 4m^2) = \exp\left(\frac{i}{t}\frac{8\mu^2}{\rho^2}\right) \sum_{r=-\infty}^{+\infty} (-i)^r J_r\left(\frac{8\mu^2}{t\rho^2}\right) \exp \frac{it}{\kappa^2}[(p+rk)^2 - 4\kappa_*^2]$$

we observe that the points

$$(p+rk)^2 = 4\kappa_*^2$$

correspond to singularities of the functions $K^{(i)}$: all derivatives with respect to p^2 beyond a particular one are divergent at these points, since the integral diverges at the upper limit. The corresponding contributions appear with a factor $(\epsilon^2 a^2)^l$, where $l = |r|, |r| \pm 2$.

In the absence of the laser field the second-order vacuum polarization exhibits a single threshold singularity at $p^2 = 4\kappa^2$ and is real for lower values of p^2. In the presence of the field the situation is entirely different: the mass becomes replaced by κ_*^2 and, starting from $p^2 = 4\kappa_*^2$ in either direction, we find a sequence of equally spaced thresholds (distance $2|pk|$), multiplied with ascending powers of $\epsilon^2 a^2$, so that the function is always complex. This is precisely the behaviour to be expected from Cutkosky's theorem applied to diagrams of the type given in figure 1, corresponding to the net emission or absorption of n laser quanta. For very high n, the singularities are, of course, unphysical; if too many quanta are absorbed, the laser beam is depleted and the description of the laser field as a classical, external field (on which our calculation is based) is no longer allowed.

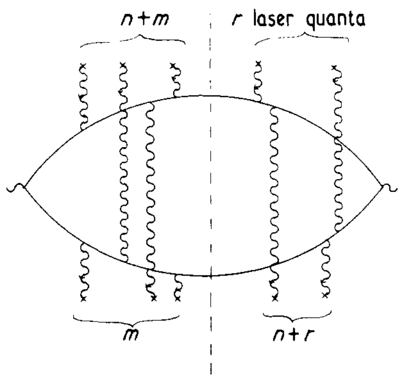

Figure 1. A typical diagram contributing to vacuum polarization in a laser field. The vertical wavy lines designate the interaction with the laser field. The indicated cut gives rise to a threshold at $(p+nk)^2 = 4\kappa^2$ which is shifted to $(p+nk)^2 = 4\kappa_*^2$ if the contributions from all values of m and r are summed.

It should be clear from this argument that the details of the analytical structure depend on the behaviour of m^2 for large argument ζ. For a finite laser pulse m^2 differs from zero only in a finite region in ζ. The quantities B in equation (4.23) as well as m^2 depend, however, also on u' in this case and it is quite unclear whether and how the notion of an 'effective' mass κ_* will emerge in the analytical structure.

6. Dispersion properties

The propagator given in § 5 contains various information on the physics of our medium, which has to be extracted in special applications. We shall discuss here only the implications for wave fields, which can be deduced from the solution (3.13) of Maxwell's equations. The integral in equation (3.13) is trivial for the laser field (2.11). The result is, using (5.7),

$$\tilde{A}_\mu(p) = \frac{1}{p^2} \sum_{i=2,3} [f_\mu h^{(i)}(p)\beta_-^{(i)}(p)\delta(1+\epsilon^2\kappa_i(p)) + f_\mu^* h^{(i)}(p_-)\beta_+^{(i)}(p_-)\delta(1+\epsilon^2\kappa_i(p_-))]. \tag{6.1}$$

We observe that in principle both excitations $i = 2, 3$ contribute to each helicity part, no matter how we choose $h_i(p)$, and that the modes are non-monochromatic, as is usual in nonlinear optics. In practice the corresponding 'mixing' is, however, very small: if we notice that the argument p is restricted by the δ functions in (6.1), we infer from relation (5.8) that $\beta_-^{(i)}(p)$ and $\beta_+^{(i)}(p)$ are of the same order of magnitude. The factor of $(p^2)^{-1}$ in front of the right-hand side of (6.1), however, has the consequence that the first term (proportional to f_μ) if $i = 3$, and the second term (proportional to f_μ^*) if $i = 2$, are of order ϵ^{-2} relative to the other one, respectively. Thus we have approximately two monochromatic modes with different dispersion laws and opposite helicity. According to the dispersion law (5.12) we may define, as usual, refractive indices n and absorption coefficients by

$$p^2 = \frac{\omega_\mathrm{p}^2}{c^2} - \boldsymbol{p}^2 = \frac{\omega_\mathrm{p}^2}{c^2}[1-(n+\mathrm{i}\varphi)^2].$$

The deviation of the refractive index from unity, $\delta = n^2-1$, and the linear absorption coefficient $\gamma = 2n\varphi\omega_\mathrm{p}/c$ become to order ϵ^2

$$\delta_{2,3} = -\left(\frac{\epsilon c}{\omega_\mathrm{p}}\right)^2 \mathrm{Re}\, N^{(-,+)}(0), \qquad \gamma_{2,3} = -\frac{\epsilon^2 c}{\omega_\mathrm{p}} \mathrm{Im}\, N^{(-,+)}(0). \tag{6.2}$$

Thus the medium shows, within this approximate picture, polar birefringence and the two helicity components are absorbed differently.

We shall now discuss the numerical behaviour of the functions

$$N^{(\pm)}(0) = K^{(2)}(0) + K^{(3)}(0) \pm \mathrm{i}\,\mathrm{sgn}(pk) K^{(4)}(0) \tag{6.3}$$

which depend on the variable ρ, equation (4.21) and the intensity parameter ν^2. Because of the oscillations of the integrand in the basic representations (4.18) and (4.23) the numerical evaluation is not without problems. The following method has proved to be useful: we extract from $K^{(i)}$ a term in which m^2 is replaced by its asymptotic value κ_*^2, writing

$$K^{(i)} = K^{(i)}(m^2 \to \kappa_*^2) + (K^{(i)}(m^2) - K^{(i)}(m^2 \to \kappa_*^2)) = K_*^{(i)} + K_1^{(i)}.$$

Using now the representations (4.23) we evaluate $K_1^{(i)}$ by numerical integration up to a fixed value in t. The remaining integral can be done analytically, using the asymptotic expansions for the Hankel functions (clearly this method does not work for too small values of ρ). The other term $K_*^{(i)}$ can be evaluated either from (4.23) using the series expansions (A1.7) and integrating analytically (this method works for $\rho \lesssim 3$), or from (4.18) by analytic integration on ζ and numerical integration on y. $K_*^{(i)}(p^2)$ exhibits a similar structure to the free vacuum polarization: there are thresholds at $p^2 = 4\kappa_*^2$ and $(p \pm k)^2 = 4\kappa_*^2$ and the functions are, respectively, real ($i = 2, 3$) and purely imaginary ($i = 4$) below the lowest threshold.

We have restricted our attention to values of the intensity parameter between 0·1 and 1, which could possibly be reached with the strongest laser available at present or in the near future. Much higher values are hardly conceivable, and for small values of ν^2 one could use the well known results of perturbation theory (Euler 1936, Heisenberg and Euler 1936, Weisskopf 1936, Karplus and Neumann 1950, 1951, Schwinger 1951).

For values of ρ up to 1 the first two terms in equation (6.3) dominate the real part. The simple power laws

$$\frac{1}{\kappa^2}\,\mathrm{Re}\,(K^{(2)}+K^{(3)}) \simeq -7{\cdot}8\times 10^{-4}\nu^2\rho^2, \qquad \frac{1}{\kappa^2}\,\mathrm{Re}\,(\mathrm{i}K^{(4)}) \simeq 4{\cdot}3\times 10^{-5}\nu^2\rho^3 \tag{6.4}$$

give a very good approximation for $\rho \lesssim 0{\cdot}5$ and a fair one up to $\rho \sim 1$. The relative difference $\mathrm{Re}(N^{(+)}(0)-N^{(-)}(0))/\mathrm{Re}\,N^{(+)}(0)$ increases linearly with ρ but is still small ($\sim 0{\cdot}06$) at $\rho = 1$. Thus we conclude that in this region the square box diagram (ie the lowest approximation to π in an expansion in powers of ν^2) gives the main contribution to the indices of refraction, even for $\nu^2 \sim 1$. As a further consequence the deviation of the refractive index from unity does not depend on the frequency ω_p of the incoming photon (as long as ρ is not too large). If its direction is nearly antiparallel to the laser, we obtain for a laser wavelength λ of $1{\cdot}06\times 10^{-6}$ m (Neodym-glass laser) and $\nu^2 = 1$ the extremely small value of

$$\delta_3 \simeq 7{\cdot}3\times 10^{-15}; \tag{6.5}$$

δ_2 has approximately the same value. That these values are so small is mainly due to the fact that δ is proportional to $(\lambda\kappa)^{-2}$. The effects could be made visible only by constructing high-intensity lasers in the x-ray region.

The imaginary part of (6.3) can be calculated with this method only for $\rho \gtrsim 0{\cdot}5$ with limited numerical effort. Here no simple power laws like (6.4) have emerged, so that in contrast to δ the absorption coefficients depend on the frequency of the incoming photon. Note that the box diagram does not contribute to the imaginary part below threshold! The imaginary part decreases rapidly with decreasing ρ and ν^2, and is smaller by two orders of magnitude than the real part at $\rho \sim 0{\cdot}6$. Here the difference between the (+) and (−) parts is rather independent of ρ and ν^2.

We find under the same conditions as were used for (6.5):

$$\begin{aligned} &\gamma_2 \simeq 0{\cdot}0026\ \mathrm{cm}^{-1}, \qquad \gamma_3 \simeq 0{\cdot}0019\ \mathrm{cm}^{-1} \qquad \text{for } \rho = 0{\cdot}5 \\ &\gamma_2 \simeq 0{\cdot}26\ \mathrm{cm}^{-1}, \qquad \gamma_3 \simeq 0{\cdot}19\ \mathrm{cm}^{-1} \qquad \text{for } \rho = 1 \end{aligned} \tag{6.6}$$

(these values of ρ correspond to photon energies of 25 and 50 GeV respectively). In general γ_3 is smaller than γ_2 for not too high values of ρ.

The reason for the rapid variability can be found in the analytic structure of the function $K^{(i)}$ as discussed in § 5. It has been mentioned that these functions have cusps at $(p \pm rk)^2 = 4\kappa_*^2$. This threshold condition reduces for $p^2 = 0$ to

$$r\rho = 4(1+\nu^2) \qquad r = 0, 1, 2, \ldots . \tag{6.7}$$

The minimum number r_0 of laser quanta which must be absorbed in order to render pair production possible increases as ρ decreases at constant ν^2. Whenever ρ passes one of the thresholds (which are more and more densely spaced in ρ), the corresponding contribution vanishes. This connection has been noticed by Narozhnyi *et al* (1964) who calculated the cross section for pair production. The corresponding results agree with those derivable from the imaginary part as computed here.

It is this threshold behaviour which is characteristic of deviations from perturbation theory: to any finite order in ν^2 the thresholds would be determined by the bare mass κ^2 rather than by the effective mass κ_*^2. Unfortunately the first and second thresholds, which are situated at $\rho = 5$ and $2{\cdot}5$ for $\nu^2 = 0{\cdot}25$, correspond to extremely high energies: for the conditions used above we need 250 GeV and 125 GeV photons respectively, and at these energies all kinds of strong interaction events would occur in experiments. Again lasers in the x-ray region could lead to visible effects at moderate energies. The preceding arguments suggest that, if successful experiments on radiative corrections enhanced by laser fields can be done at all, they cannot be described by a constant crossed field. Even at low energies one has to be careful: the level splitting given by (6.4) is proportional to ρ^3 and therefore does not agree with that calculated in a constant crossed field (Ritus 1969).

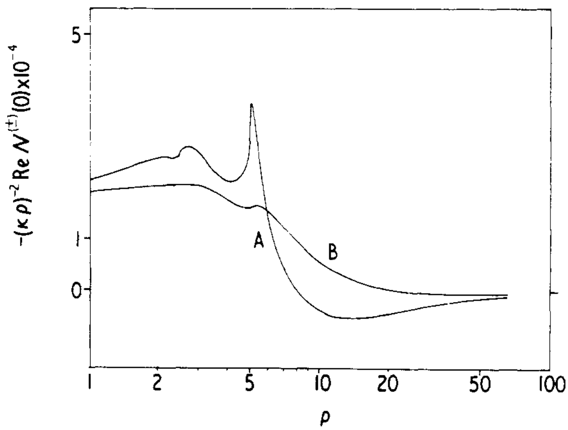

Figure 2. The functions $-(\kappa\rho)^{-2}$ Re $N^{(\pm)}(0)$ which are proportional to the deviation δ of the refractive index from unity ($\nu^2 = 0{\cdot}25$; A: $N^+(0)$; B: $N^-(0)$).

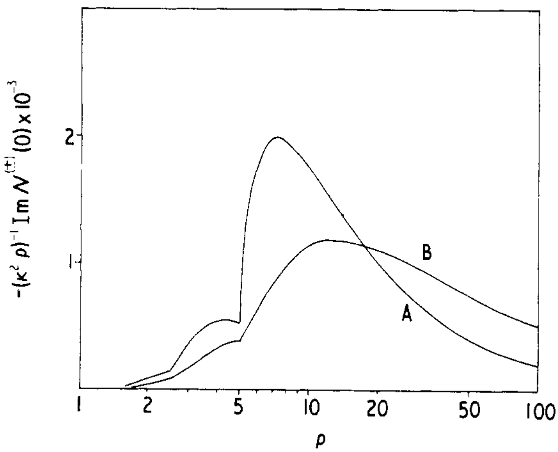

Figure 3. The functions $-(\kappa^2\rho)^{-1}$ Im $N^{(\pm)}(0)$ which are proportional to the linear absorption coefficient γ($\nu^2 = 0{\cdot}25$; A: $N^+(0)$; B: $N^-(0)$).

At present the dispersion curves given in figures 2 and 3 serve only as illustrations of the theory. The first two thresholds are quite obvious and we observe precisely the behaviour known from conventional optics. This is dictated by causality as expressed in the Kramers–Kronig dispersion relations. In particular we observe that the complex index of refraction is in the upper half plane, the dispersion is anomalous in regions of strong absorption and δ passes zero at about the maximum of absorption.

Acknowledgments

We are indebted to Dr H Ungerer for extensive help with the numerical calculations and to Dr P Breitenlohner and Professor F Rohrlich for discussions.

Appendix 1

For most calculations involving particles in laser fields the electron propagator G is needed. From several possible representations (cf Mitter 1975), one which looks at first glance rather complicated (M 4.3, 13) has proved to be the most useful for practical calculations. This representation contains a number of functions of the external field which show up in all expressions derived from G, eg in vacuum polarization (cf (4.14)). Here we shall state some properties of these functions which are useful, if not substantial, for all corresponding calculations. The functions depend on two variables ξ, ξ' or $\eta = \frac{1}{2}(\xi+\xi')$, $\zeta = (\xi-\xi')$ and are constructed from the building blocks

$$a_i(\xi), \qquad b_i(\xi,\xi') = \int_{\xi'}^{\xi} \mathrm{d}\bar{\xi}\, a_i(\bar{\xi}), \qquad \alpha(\xi,\xi') = \int_{\xi'}^{\xi} \mathrm{d}\bar{\xi}\, a_i(\bar{\xi}) a_i(\bar{\xi}) \tag{A1.1}$$

in the following way:

$$\begin{aligned}
T &= \frac{\epsilon^2 a^2}{\zeta}(\alpha - b_i b_i) \\
M_i &= -\frac{\epsilon a}{2\zeta}(a_i(\xi) - a_i(\xi')) \\
L_i &= \frac{\epsilon a}{2\zeta}\left(a_i(\xi) + a_i(\xi') - \frac{2}{\xi} b_i\right) \\
N &= \zeta(L_i L_i + M_i M_i) - T/\zeta \\
R &= N - 2\zeta M_i M_i \\
\hat{R} &= -2\mathrm{i}\zeta L_i \epsilon_{ij} M_j .
\end{aligned} \tag{A1.2}$$

We have the following relations:

$$\frac{\partial T}{\partial \zeta} = N, \qquad \frac{1}{2}\frac{\partial T}{\partial \eta} = -2\zeta^2 L_i M_i, \qquad \frac{1}{4}\frac{\partial^2 T}{\partial \eta^2} = \frac{\partial^2 T}{\partial \zeta^2} + 4 L_i L_i - \frac{2T}{\zeta^2}. \tag{A1.3}$$

Furthermore we note that

$T, M_i, \hat{R}$ are symmetric in ξ

N, L_i, R are antisymmetric in ξ

and that all functions except M_i vanish for $\zeta = 0$. For a plane wave train of infinite extent and circular polarization (2.11) we have

$$(M_1, M_2) = (\sin\eta, \cos\eta)M(\zeta), \qquad (L_1, L_2) = (-\cos\eta, \sin\eta)L(\zeta) \qquad \text{(A1.4)}$$

where the magnitudes L, M of these two-vectors can be expressed in terms of the first two spherical Bessel functions

$$\mathrm{j}_0\left(\frac{\zeta}{2}\right) = \left(\frac{\pi}{\zeta}\right)^{1/2} J_{1/2}\left(\frac{\zeta}{2}\right) = \frac{2}{\zeta}\sin\frac{\zeta}{2}$$

$$\mathrm{j}_1\left(\frac{\zeta}{2}\right) = \left(\frac{\pi}{\zeta}\right)^{1/2} J_{3/2}\left(\frac{\zeta}{2}\right) = \frac{4}{\zeta^2}\sin\frac{\zeta}{2} - \frac{2}{\zeta}\cos\frac{\zeta}{2}$$

in the following way

$$M = \frac{\epsilon a}{2}\mathrm{j}_0\left(\frac{\zeta}{2}\right), \qquad L = \frac{\epsilon a}{2}\mathrm{j}_1\left(\frac{\zeta}{2}\right) = -2\frac{\mathrm{d}}{\mathrm{d}\zeta}M.$$

The matrix K_{ij}, equation (4.15*a*), becomes

$$K = \begin{pmatrix} \cos\eta, & -\sin\eta\cos\eta \\ -\sin\eta\cos\eta, & \sin^2\eta \end{pmatrix}(L^2+M^2). \qquad \text{(A1.5)}$$

Some additional relations are

$$L_iM_i = 0, \qquad T = \epsilon^2a^2 - 4M^2, \qquad R = 2M(2L-\zeta M) \qquad \text{(A1.6)}$$

$$N = 4LM, \qquad \hat{R} = 2i\zeta LM.$$

Power series expansions may be obtained from

$$LM = \epsilon^2a^2\sum_{r=0}^{\infty}\frac{(-1)^r(r+1)}{(2r+4)!}\zeta^{2r+1}$$

$$M^2 = \frac{\epsilon^2a^2}{2}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+2)!}\zeta^{2r} \qquad \text{(A1.7)}$$

$$L^2 = \frac{\epsilon^2a^2}{2}\sum_{r=0}^{\infty}\frac{(-1)^r(r+1)}{(r+3)(2r+4)!}\zeta^{2r+2}$$

Appendix 2

A monochromatic circularly polarized plane wave field is invariant with respect to the following operation: translation by an arbitrary vector a^μ followed by a spatial rotation about the direction $\boldsymbol{k}$ of the field of angle $k_\mu a^\mu$. Let us apply the corresponding unitary transformation to the photon propagator in the laser field $D'_{\mu\nu}(x, x')$ and choose

$$a^\mu = -\tfrac{1}{2}(x+x')^\mu = -X^\mu.$$

Due to invariance of the vacuum with respect to this operation we then have

$$\begin{aligned}
\mathrm{i}\langle 0_+|0_-\rangle D'_{\mu\nu}(x,x') &= \langle 0_+|TA_\mu(x)A_\nu(x')|0_-\rangle \\
&= \Lambda^{-1}(-X)_\mu{}^\kappa \Lambda^{-1}(-X)_\nu{}^\lambda \\
&\quad \times \langle 0_+|TA_\kappa(\tfrac{1}{2}\Lambda(-X)(x-x'))A_\lambda(-\tfrac{1}{2}\Lambda(-X)(x-x'))|0_-\rangle.
\end{aligned} \tag{A2.1}$$

If we have the wavevector $\boldsymbol{k}$ in the z direction, $\Lambda(a)$ is essentially the rotation matrix

$$\Lambda(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos ka & \sin ka & 0 \\ 0 & -\sin ka & \cos ka & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A2.2}$$

and $\Lambda^{-1}(a) = \Lambda(-a)$. Due to relativistic and gauge invariance the matrix element on the right-hand side of (A2.1) depends only on the scalars $(\Lambda(-X)(x-x'))^2 = (x-x')^2$ and $k\Lambda(-X)(x-x') = k(x-x')$ which are both independent of X. From that and (A2.1, 2) we conclude that the longitudinal and time-like components D'_{33}, D'_{00}, D'_{30}, D'_{03} are independent of X and, in momentum space, become proportional to $\delta(p-p')$. As to the transverse left- or right-handed components $A^{(\pm)} = A_1 \pm \mathrm{i}A_2$, we find

$$\langle 0_+|TA^\pm(x)A^\pm(x')|0_-\rangle = \mathrm{e}^{\mp 2\mathrm{i}kX}\langle 0_+|TA^\pm(\tfrac{1}{2}\Lambda(-X)(x-x'))A^\pm(-\tfrac{1}{2}\Lambda(-X)(x-x'))|0_-\rangle$$

$$\langle 0_+|TA^\pm(x)A^\mp(x')|0_-\rangle = \langle 0_+|TA^\pm(\tfrac{1}{2}\Lambda(-X)(x-x'))A^\mp(-\tfrac{1}{2}\Lambda(-X)(x-x'))|0_-\rangle$$

and conclude in the same way that the components $D'_{\pm\mp} = \langle 0_+|TA^{\mp *}A^\mp|0_-\rangle$ which preserve the left- or right-handedness become diagonal in momentum space whereas those which change it become proportional to $\delta(p-p'\pm 2k)$. The mixed components $D'_{3\pm}, D'_{\pm 0}$ etc become by the same argument proportional to $\delta(p-p'\pm k)$. As this is forbidden by Furry's theorem they have to vanish, which we already know from the tensor decomposition (4.5).

So we have established the general structure of (5.9) as to be valid to all orders of ϵ^2.

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